THE CONJUGACY LOCUS OF CAYLEY-SALMON LINES

JAYDEEP CHIPALKATTI

ABSTRACT: Given six points on a conic, Pascal's theorem gives rise to a well-known configuration called the *hexagrammum mysticum*. It consists of, amongst other things, twenty Steiner points and twenty Cayley-Salmon lines. It is a classical theorem due to von Staudt that the Steiner points fall into ten conjugate pairs with reference to the conic; but this is not true of the C-S lines for a general choice of six points. It is shown in this paper that the C-S lines are pairwise conjugate precisely when the original sextuple is *tri-symmetric*. The variety of tri-symmetric sextuples turns out to be arithmetically Cohen-Macaulay of codimension two. We determine its SL_2-equivariant minimal resolution.

CONTENTS

1. INTRODUCTION

The main result of this paper involves the so-called Caylay-Salmon lines, which form a subconfiguration of the famous *hexagrammum mysticum* of Pascal. I have retained the notation of my earlier paper [4], but some of the background is reproduced below for ease of reading.

1.1. **Pascal lines.** Let $\mathcal{K}$ denote a nonsingular conic in the complex projective plane. Consider six distinct points $\Gamma = \{A, B, C, D, E, F\}$ on $\mathcal{K}$, arranged as an array $\begin{bmatrix} A & B & C \\ F & E & D \end{bmatrix}$. Pascal's theorem says that the three cross-hair intersection points
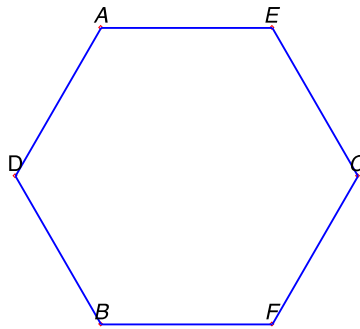
$$AE \cap BF, \quad AD \cap CF, \quad BD \cap CE$$

(corresponding to the three minors of the array) are collinear. The line containing them (usually called the Pascal line, or just the Pascal) is denoted by $\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}$. It appears as $k(1, 23)$ in the usual labelling schema for Pascals (see [4, §2]). Similarly, the lines $k(2, 13)$ and $k(3, 12)$ are respectively equal to

$$\left\{ \begin{matrix} A & B & C \\ D & F & E \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} A & B & C \\ E & D & F \end{matrix} \right\}.$$

The labelling schema for Pascals relies upon the unique outer automorphism of the symmetric group on six objects. It can be extended to all geometric elements in the *hexagrammum mysticum*. Its advantage is that the incidence relations between them look far more natural when expressed in this schema.

1.2. Pascal's theorem can also be rephrased as saying that the three pairs of opposite sides of a hexagon inscribed in $\mathcal{K}$ meet in collinear points. Indeed, the intersections of the opposite sides of the hexagon

are exactly the three cross-hair intersection points of $\begin{bmatrix} A & B & C \\ F & E & D \end{bmatrix}$. Thus both versions of the theorem are easily interchangeable.

1.3. Steiner Points. It was proved by Steiner that the lines $k(1,23), k(2,13), k(3,12)$ are concurrent, and their common point of intersection is called a Steiner point, denoted by[1] $G[123]$. This is understood to hold for any three indices in the same pattern. If SIX denotes the set $\{1, 2, \ldots, 6\}$, then for any three indices $x, y, z \in$ SIX, the Pascals

$$k(x, yz), \quad k(y, xz), \quad k(z, xy),$$

intersect in the point $G[xyz]$, giving altogether $\binom{6}{3} = 20$ Steiner points. This theorem, together with similar incidence theorems mentioned below may be found in Salmon's [18, Notes], which also contains references to earlier literature on the subject. Salmon does not use the k-notation however, but there is a detailed explanation of it in Baker's [2, Note II]. A short modern account of the *hexagrammum mysticum* is given by Conway and Ryba [6].

1.4. Kirkman points and Cayley-Salmon lines. It is a theorem due to Kirkman that the lines

$$k(1,23), \quad k(1,24), \quad k(1,34)$$

are concurrent, and their common point of intersection is called the Kirkman point $K[1, 234]$. There are 60 such points $K[w, xyz]$, corresponding to indices $w \in$ SIX, $\{x, y, z\} \subseteq$ SIX $\setminus \{w\}$. It was proved by Cayley and Salmon that the Kirkman points

$$K[4, 123], \quad K[5, 123], \quad K[6, 123]$$

are collinear, and the common line containing them is called the Cayley-Salmon line[2] $g(123)$. There are 20 such lines $g(xyz)$ for $\{x, y, z\} \subseteq$ SIX. It is also the case that $g(xyz)$ contains the Steiner point $G[xyz]$.

For a *general* choice of six points Γ, all of the lines and points above are pairwise distinct, and there are no further incidences apart from those already mentioned. (It may happen that some of the lines and points become undefined for special positions of Γ; this will be explained as and when necessary.) Using the standard notation for incidence correspondences (see [13]), the situation for a general Γ can be summarised as follows:

- The K-points and g-lines make a $(60_1, 20_3)$ configuration. That is to say, each of the 60 K-points lies on one g-line, and each of the 20 g-lines contains three K-points.
- The K-points and k-lines make a $(60_3, 60_3)$ configuration.
- The G-points and k-lines make a $(20_3, 60_1)$ configuration.
- The G-points and g-lines make a $(20_1, 20_1)$ configuration.

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[1]It is a notational convention that 123 stands for the set $\{1, 2, 3\}$, and hence the order of indices is irrelevant. This applies to all similar situations, so that $k(1, 23)$ is the same as $k(1, 32)$ etc.

[2]The illustrious 27 lines on a nonsigular cubic surface (see [12, Ch. 4]) are also sometimes called the Cayley-Salmon lines. Although our context is different, there is, in fact, a thematic connection between the geometry of the 27 lines and the *hexagrammum mysticum* (see [17]).

We have throughout used uppercase letters for points, and lowercase letters for lines. The governing pattern is that, if the A-points and b-lines form an (r_s, t_u) configuration, then the B-points and a-lines form a (t_u, r_s) configuration.[3] It is tempting to conjecture that this numerical duality should be explainable as a pole-polar duality, but this is surprisingly not so (cf. [14, p. 194]). The following pair of facts signals a breakdown in symmetry. Assume Γ to be general, and let $\{u, v, w, x, y, z\} = \text{SIX}$. Then

- The points $G[uvw]$ and $G[xyz]$ are conjugate with respect to $\mathcal{K}$; that is to say, the polar line of $G[uvw]$ passes through $G[xyz]$ and conversely. This is a theorem due to von Staudt (see [21] or [16, §86.2]).
- The lines $g(uvw), g(xyz)$ are not conjugate; that is to say, the pole of $g(uvw)$ does not lie on $g(xyz)$. This is easily checked with an example; see §2.2 below.

It is natural to ask whether the g-lines are in fact pairwise conjugate for *special* positions of Γ. The main result of this paper answers this question.

1.5. The tri-symmetric configuration. Recall that $\mathcal{K}$ is isomorphic to the projective line $\mathbb{P}^1$. We will say that a sextuple Γ is *tri-symmetric*, if it is projectively equivalent to

$$\left\{ 0, \ 1, \ \infty, \ p, \ \frac{p-1}{p}, \ \frac{1}{1-p} \right\}, \tag{1.1}$$

for some $p \in \mathbb{P}^1$. The rationale behind this name will be clarified in §2.4. The configuration was classically known in a different guise (see §5.5), but it arose in [4, §4.5] in the course of solving a different problem.

Since the points are assumed distinct,

$$p \neq 0, \ 1, \ \infty, \ \frac{1 \pm \sqrt{-3}}{2}.$$

It may happen that the three Kirkman points $K[w, xyz], w \in \text{SIX} \setminus \{x, y, z\}$ all coincide, so that the line $g(xyz)$ becomes undefined. A complete list of such cases is given in §4.9.

1.6. We will say that a sextuple Γ satisfies Cayley-Salmon conjugacy (CSC), if the following property holds: The lines $g(uvw)$ and $g(xyz)$ are conjugate, whenever both are defined and $\{u, v, w, x, y, z\} = \text{SIX}$.

This is the main result of the paper:

Theorem 1.1. The following statements are equivalent:

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[3]There are an additional fifteen i-lines and fifteen I-points which also obey this pattern, but we will not pursue this digression. See the notes by Baker and Salmon referred to above.

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(1) Γ is tri-symmetric.

(2) Γ satisfies CSC.

Let Ω denote the Zariski closure of the subset of tri-symmetric sextuples inside

$$\mathrm{Sym}^6 \mathcal{K} \simeq \mathrm{Sym}^6 \mathbb{P}^1 \simeq \mathbb{P}^6.$$

It is a four-dimensional irreducible projective subvariety. (The p contributes one parameter, and the additional three come from SL_2 acting on $\mathbb{P}^1$.) In §3, we will determine the group of symmetries of a generic tri-symmetric sextuple and use it to calculate the degree of Ω. The main theorem is proved in §4; it relies upon the properties of a collection of invariant polynomials associated to 3-element subsets of SIX. These properties were discovered by explicitly calculating and factoring these polynomials in MAPLE.

The SL_2-equivariant minimal resolution of the ideal of Ω is determined in the concluding section. It implies that Ω is an arithmetically Cohen-Macaulay subvariety of $\mathbb{P}^6$.

2. PRELIMINARIES

2.1. The base field is throughout **C**. As in [4, §3], let S_d denote the vector space of homogeneous polynomials of degree d in the variables $\{x_1, x_2\}$. We identify the projective plane with $\mathbb{P}S_2$, and $\mathcal{K}$ with the image of the Veronese imbedding $\mathbb{P}S_1 \longrightarrow \mathbb{P}S_2$. A nonzero quadratic form in the x_i simultaneously represents a point in $\mathbb{P}^2$ and its polar line with respect to $\mathcal{K}$. All the line-intersections or point-joins can then be calculated as transvectants of binary forms. In particular, two lines represented by quadratic forms λ and μ are conjugate exactly when the transvectant $(\lambda, \mu)_2$ is zero.

The sextuple $\Gamma = \{z_1, \ldots, z_6\} \subseteq \mathbf{C} \cup \{\infty\} = \mathbb{P}^1$ is identified with the sextic form $\prod_{i=1}^{6} (x_1 - z_i \, x_2)$. (If $z_i = \infty$, we interpret the corresponding factor as x_2.) In this way, Γ can be seen as a point of $\mathbb{P}S_6 \simeq \mathbb{P}^6$.

2.2. For instance, consider the sextuple

$$A = 0, \quad B = 1, \quad C = \infty, \quad D = -1, \quad E = 3, \quad F = -5.$$

Then E is represented by $(x_1 - 3\,x_2)^2$, and C by x_2^2 etc. Then the line EC corresponds to

$$((x_1 - 3\,x_2)^2, x_2^2)_1 = x_2 \, (x_1 - 3\,x_2),$$

and one can calculate any required line or point from §1. In particular, $g(123)$ is given by the form $3\,x_1^2 - 26\,x_1\,x_2 - 5\,x_2^2$. All the other g-lines are calculated similarly, and then it is straightforward to check that

$$(g(123), g(\alpha))_2 \neq 0,$$

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for any 3-element subset $\alpha \subseteq$ SIX. That is to say, $g(123)$ is not conjugate to any C-S line. Since CSC is a closed condition on sextuples, it follows that a general sextuple does not satisfy CSC.

2.3. Recall that the cross-ratio of four ordered points on $\mathbb{P}^1$ is defined to be

$$\langle z_1, z_2, z_3, z_4 \rangle = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

If r denotes the original cross-ratio, then permuting the points in all possible ways gives six variants (see [19, Ch. 1]), namely

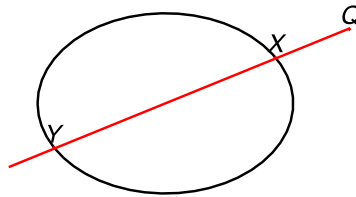$$r, \quad \frac{1}{r}, \quad 1 - r, \quad \frac{1}{1 - r}, \quad \frac{r - 1}{r}, \quad \frac{r}{r - 1}.$$

One can reformulate the notion of tri-symmetry in terms of cross-ratios. As in [4, §3], consider the set of letters LTR $= \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}\}$, and define a *hexad* to be an injective map LTR $\xrightarrow{h} \mathcal{K}$. Write $h(\mathbb{A}) = A, h(\mathbb{B}) = B$, and so on for the corresponding distinct points on $\mathcal{K}$. Then a sextuple Γ is tri-symmetric, if and only if there exists a hexad with image Γ such that

$$\langle A, B, C, F \rangle = \langle B, C, A, E \rangle = \langle C, A, B, D \rangle. \tag{2.1}$$

Indeed, if we let $A = 0, B = 1, C = \infty$, and if p denotes this common cross-ratio, then

$$D = p, \quad E = \frac{p - 1}{p}, \quad F = \frac{1}{1 - p}. \tag{2.2}$$

2.4. Recall ([20, Ch. VI]) that a point Q in $\mathbb{P}^2 \setminus \mathcal{K}$ defines an involution σ_Q on $\mathcal{K}$, which sends a point X to the other intersection point of $\overline{QX}$ with $\mathcal{K}$.

$$\sigma_Q : X \to Y$$

Now a sextuple Γ is said to be in involution (see Figure 1), if it is invariant with respect to σ_Q for some Q. In other words, there should exist a point $Q \in \mathbb{P}^2 \setminus \mathcal{K}$ and three lines L, L', L'' through Q such that $\Gamma = \mathcal{K} \cap (L \cup L' \cup L'')$. One says that Q is a centre of involution for Γ.

Now a tri-symmetric sextuple has the peculiarity that it has three such centres, which are in fact collinear. Observe that in Figure 2 on page 8, each of the three points Q_4, Q_5, Q_6 is a centre of involution in such a way that
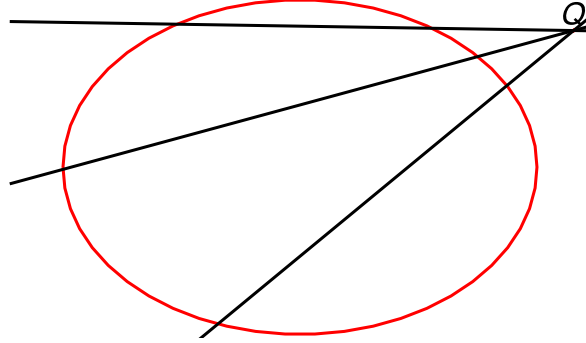
- AE, CD, BF intersect in Q_4,

FIGURE 1. A sextuple in involution

- AF, CE, BD intersect in Q_5,
- AD, BE, CF intersect in Q_6.

The Pascals associated to a tri-symmetric sextuple are discussed at length in [4, §4.5], where the co-ordinates of the Q_i are also explicitly written down. The following lemma is thematically relevant, although not logically necessary for what follows.

Lemma 2.1. Assume that Γ admits exactly three centres of involution Q_u, Q_v, Q_w, and more-over the corresponding involutions are mutually non-commuting. Then the three centres must be collinear.

PROOF. Since $\sigma_{Q_u} \sigma_{Q_v} \sigma_{Q_u}$ is a non-identity involution on $\mathcal{K}$, it must equal σ_{Q_w}. Now assume that $\overline{Q_u Q_v} \cap \mathcal{K} = \{R, R'\}$. Then $\{R, R'\}$ is an orbit of $\sigma_{Q_u}, \sigma_{Q_v}$ and hence also of σ_{Q_w}. But then it follows that Q_w lies on the line $\overline{RR'} = \overline{Q_u Q_v}$. The reader should check that the proof is valid with small modifications if R, R' coincide. □

A general tri-symmetric sextuple does have exactly three collinear centres of involution as in the lemma. It may however, specialise to a configuration with more than three centres which are no longer collinear (see section 5.3).

2.5. The outer isomorphism. It is well-known that the permutation group on six objects has a unique outer automorphism (see [8]). Part of its charm is that it is implicated everywhere in the geometry of the *hexagrammmum mysticum*. We will use it in the following version: let $\mathfrak{S}(X)$ denote
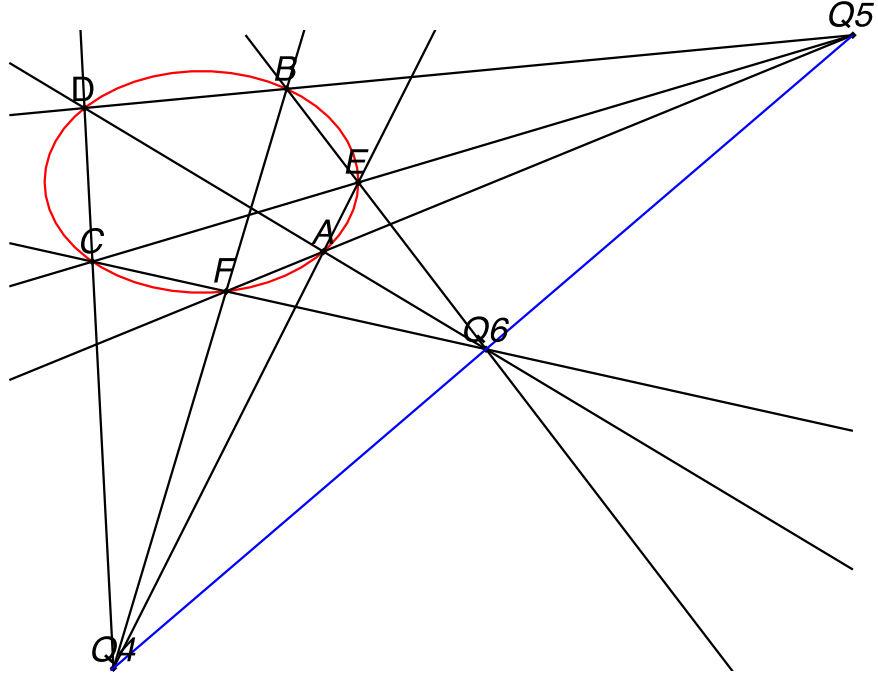
FIGURE 2. A tri-symmetric sextuple

the permutation group on the set X. Then the following symmetric table defines an isomorphism $\mathfrak{S}(\text{LTR}) \xrightarrow{\zeta} \mathfrak{S}(\text{SIX})$.

	$\mathbb{A}$	$\mathbb{B}$	$\mathbb{C}$	$\mathbb{D}$	$\mathbb{E}$	$\mathbb{F}$
$\mathbb{A}$		14.25.36	16.24.35	13.26.45	12.34.56	15.23.46
$\mathbb{B}$	14.25.36		15.26.34	12.35.46	16.23.45	13.24.56
$\mathbb{C}$	16.24.35	15.26.34		14.23.56	13.25.46	12.36.45
$\mathbb{D}$	13.26.45	12.35.46	14.23.56		15.24.36	16.25.34
$\mathbb{E}$	12.34.56	16.23.45	13.25.46	15.24.36		14.26.35
$\mathbb{F}$	15.23.46	13.24.56	12.36.45	16.25.34	14.26.35	

For instance, the entry in row $\mathbb{A}$ and column $\mathbb{B}$ is 14.25.36, and hence ζ takes the transposition $(\mathbb{A}\,\mathbb{B})$ to the element $(1\,4)\,(2\,5)\,(3\,6)$ of cycle type $2 + 2 + 2$. In the reverse direction, ζ^{-1} takes $(1\,2)$ to $(\mathbb{A}\,\mathbb{E})\,(\mathbb{B}\,\mathbb{D})\,(\mathbb{C}\,\mathbb{F})$, because these are precisely the positions in the table where the pair 12 appears. The action of ζ or ζ^{-1} on an arbitrary element can be read off by writing it as a product

of transpositions. The construction of this table can be traced to J. J. Sylvester (see [23]), although the original context is rather more combinatorial than group-theoretic.

3. THE SHUFFLES OF A TRI-SYMMETRIC SEXTUPLE

3.1. Let Γ be a tri-symmetric sextuple. We will say that a hexad $\text{LTR} \xrightarrow{h} \Gamma$ is an *alignment* if equations (2.1) hold. It would be convenient to characterise all shuffles of LTR which preserve this property. After an automorphism of $\mathbb{P}^1$, we may assume that h is given by

$$\mathbb{A} \to 0, \quad \mathbb{B} \to 1, \quad \mathbb{C} \to \infty, \quad \mathbb{D} \to p, \quad \mathbb{E} \to \frac{p-1}{p}, \quad \mathbb{F} \to \frac{1}{1-p}.$$

Let $\Theta(p)$ denote the subgroup of bijections $\text{LTR} \xrightarrow{\eta} \text{LTR}$ such that

$$\langle A', B', C', F' \rangle = \langle B', C', A', E' \rangle = \langle C', A', B', D' \rangle, \tag{3.1}$$

where A' stands for $h \circ \eta(\mathbb{A})$ and similarly for B' etc. This group is completely determined by formal properties of cross-ratios for a general value of p, and as such is independent of p. We will denote the generic case simply by Θ, and proceed to determine its structure. (The group may be larger for special values of p; see §5.3 below.) This will be used later to find the degree of the locus of tri-symmetric sextuples.

With the alignment as above,

$$\langle E, F, D, B \rangle = \langle F, D, E, A \rangle = \langle D, E, F, C \rangle = p, \tag{3.2}$$

and hence $(\mathbb{A}\,\mathbb{E})\,(\mathbb{B}\,\mathbb{F})\,(\mathbb{C}\,\mathbb{D}) \in \Theta$. More generally, it is easy to verify that

$$(x\,x')\,(y\,y')\,(z\,z') \in \Theta, \tag{3.3}$$

whenever

$$\{x, y, z\} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}\}, \quad \text{and} \quad \{x', y', z'\} = \{\mathbb{D}, \mathbb{E}, \mathbb{F}\}.$$

(For instance, if $\eta = (\mathbb{A}\,\mathbb{D})\,(\mathbb{B}\,\mathbb{F})\,(\mathbb{C}\,\mathbb{E})$, then each term in (3.1) becomes $p/(p-1)$.) It immediately follows that $(x\,x')\,(y\,y') \in \Theta$ whenever $x, x' \in \{\mathbb{A}, \mathbb{B}, \mathbb{C}\}$ and $y, y' \in \{\mathbb{D}, \mathbb{E}, \mathbb{F}\}$. Moreover, since $(\mathbb{A}\,\mathbb{B})\,(\mathbb{D}\,\mathbb{E})\,(\mathbb{A}\,\mathbb{C})\,(\mathbb{D}\,\mathbb{E}) = (\mathbb{A}\,\mathbb{B}\,\mathbb{C})$ etc, every 3-cycle in either $\mathfrak{S}(\{\mathbb{A}, \mathbb{B}, \mathbb{C}\})$ or $\mathfrak{S}(\{\mathbb{D}, \mathbb{E}, \mathbb{F}\})$ is in Θ.

Proposition 3.1. For a general value of p, the elements in (3.3) generate Θ.

PROOF. Let $\Theta' \subseteq \Theta$ denote the subgroup they generate, and consider the subgroup

$$\Phi = \mathfrak{S}(\{1, 2, 3\}) \times \mathfrak{S}(\{4, 5, 6\}), \tag{3.4}$$

of $\mathfrak{S}(\text{SIX})$. We claim that there is an equality $\zeta(\Theta') = \Phi$. Indeed, the table in §2.5 shows directly that ζ takes the six elements in (3.3) to the 2-cycles $(x\,x')$, where x, x' are both in $\{1, 2, 3\}$ or in $\{4, 5, 6\}$. (For instance, $(\mathbb{A}\,\mathbb{E})\,(\mathbb{B}\,\mathbb{F})\,(\mathbb{C}\,\mathbb{D})$ goes to $(5\,6)$.) This proves the claim.

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Now assume that $\Theta' \subsetneq \Theta$, and let $z \in \zeta(\Theta) \setminus \Phi$ denote an element with the maximum number of fixed points. It must be the case that z takes elements in $\{1, 2, 3\}$ to $\{4, 5, 6\}$ and conversely; for if not, one could increase the number of fixed points by multiplying it by an element of Φ. But then after conjugation by an element of Φ, one may assume z to be one of the following: $(1\,4), (1\,4)\,(2\,5), (1\,4)\,(2\,5)\,(3\,6), (1\,4\,2\,5), (1\,4\,2\,5)\,(3\,6), (1\,4\,2\,5\,3\,6)$. Now apply ζ^{-1} to each of these, and check that none of the images belongs to Θ. This proves that $\Theta' = \Theta$. $\qquad \square$

It follows that $|\Theta| = 36$. The isomorphism $\Theta \simeq \Phi$ shows a common phenomenon surrounding the outer isomorphism; namely a complicated structure on one side of ζ often appears as a simpler structure on the other side.

3.2. Now let $\Omega \subseteq \mathbb{P}^6$ denote the Zariski closure of the set of tri-symmetric sextuples. It is an irreducible projective fourfold.

Proposition 3.2. The degree of Ω is 16.

The proof will emerge from the discussion below. If $z \in \mathcal{K}$ is an arbitrary point, then $\{\Gamma \in \mathbb{P}^6 : z \in \Gamma\}$ is a hyperplane in $\mathbb{P}^6$. Since the degree of Ω is the number of points in its intersection with four general hyperplanes, we are reduced to the following question: Given a set of four general points $Z = \{z_1, \ldots, z_4\} \subseteq \mathcal{K} \simeq \mathbb{P}^1$, find the number of tri-symmetric sextuples Γ such that $Z \subseteq \Gamma$. In other words, find all tri-symmetric extensions of Z.

Let the variables $a, b, \ldots, f$ respectively stand for the coordinates of $A, B, \ldots, F$ on $\mathcal{K}$. Then conditions (2.1) correspond to equations

$$\frac{(a-c)(b-f)}{(a-f)(b-c)} = \frac{(b-a)(c-e)}{(b-e)(c-a)} = \frac{(c-b)(a-d)}{(c-d)(a-b)}. \tag{3.5}$$

3.3. By an *assignment*, we will mean an injective map $Z \xrightarrow{u} \{a, \ldots, f\}$. Given such an assignment, say

$$z_1 \to c, \quad z_2 \to e, \quad z_3 \to a, \quad z_4 \to d,$$

we abbreviate it by $[c, e, a, d]$. Now substitute $c = z_1, \ldots, d = z_4$ into (3.5), and solve for the remaining unknowns b and f. Depending on the number of solutions so obtained, one would get one or more tri-symmetric sextuples Γ containing Z. It it clear that every extension must come from an assignment. The group Θ acts naturally on the set of assignments by permuting the $a, \ldots, f$, and two assignments which are in the same orbit have the same extensions. Given u as above, consider the integer $n(u) = \text{card}\,(\text{image}(u) \cap \{a, b, c\})$. It must be either $1, 2$ or 3.

3.4. Assume $n(u) = 3$. After an action by an element of Θ, we may assume that u is one of the following:

$$[d, a, b, c], \quad [a, d, b, c], \quad [a, b, d, c], \quad [a, b, c, d]. \tag{3.6}$$

(For example, $[c, a, f, b]$ can be changed to $[a, c, e, b]$ by $(\mathbb{A}\,\mathbb{C})\,(\mathbb{E}\,\mathbb{F})$, and then to $[a, b, d, c]$ by $(\mathbb{B}\,\mathbb{C})\,(\mathbb{D}\,\mathbb{E})$.) In each case, (3.5) give linear equations for the remaining variables e and f which determine them uniquely. For instance, if we let the z_i to be respectively $0, 1, \infty, r$, then the $[d, a, b, c]$ case gives the extension $Z \cup \left\{ \frac{r^2 - r + 1}{r}, r^2 - r + 1 \right\}$. The remaining cases in (3.6) give three more extensions by the following enlargements of Z:

$$\left\{ \frac{r^2}{r-1}, r - r^2 \right\}, \quad \left\{ \frac{r}{r^2 - r + 1}, \frac{r^2}{r^2 - r + 1} \right\}, \quad \left\{ \frac{1}{1-r}, \frac{r-1}{r} \right\}.$$

Thus altogether we get four extensions. The assumption $n(u) = 1$ gives nothing new, since we are back to $n(u) = 3$ after an action by $(\mathbb{A}\,\mathbb{D})\,(\mathbb{B}\,\mathbb{E})\,(\mathbb{C}\,\mathbb{F})$.

3.5. If $n(u) = 2$, then after an action of Θ we may assume that $\mathrm{image}(u) = \{a, b, e, f\}$. We can then reduce further to one of the following six cases:

$$[a, b, x, y], \quad [a, x, b, y], \quad [a, x, y, b], \quad \text{where } \{x, y\} = \{e, f\}.$$

(For instance, $[f, b, a, e]$ is changed to $[b, f, e, a]$ by $(\mathbb{A}\,\mathbb{E})\,(\mathbb{B}\,\mathbb{F})(\mathbb{C}\,\mathbb{D})$, and then to $[a, e, f, b]$ by $(\mathbb{A}\,\mathbb{B})\,(\mathbb{E}\,\mathbb{F})$.) Given any such u, the first pair of equations in (3.5) gives a quadratic in c, and either of its roots will determine the value of d uniquely. Thus we get $6 \times 2 = 12$ extensions altogether, and the fact that they are generically distinct is verified by a direct calculation as above. Thus $\deg \Omega = 4 + 12 = 16$. $\qquad\square$

4. THE CAYLEY-SALMON POLYNOMIALS

Consider the graded polynomial ring $\mathfrak{P} = \mathbf{C}[a, b, c, d, e, f]$. One can formulate the CSC condition as the vanishing of certain homogeneous polynomials in $\mathfrak{P}$; this will lead to a proof of the main theorem.

4.1. As indicated earlier, each of the lines and points mentioned in §1 is represented by a quadratic form in x_1 and x_2, with coefficients in $\mathfrak{P}$. For instance, the line $k(1, 23)$ appears as $\frac{1}{4}(c - d)$ times

$$(c\,f - c\,e - a\,d - b\,f + b\,d + a\,e)\,x_1^2 + \cdots + (a\,c\,e\,f - a\,c\,d\,e + b\,c\,d\,f - b\,c\,e\,f + a\,b\,d\,e - a\,b\,d\,f)\,x_2^2.$$

Since $c - d \neq 0$, we are free to ignore the extraneous multiplicative factor, and this will always be done in the sequel. Henceforth we will not write out such forms explicitly, since the expressions are generally lengthy and not much is to be learned by merely looking at them. I have programmed the entire procedure in MAPLE, and the structure of the polynomials $\mathcal{C}_\alpha$ described below was discovered in this way.

4.2. If quadratic forms Q, Q' represent the lines $g(123), g(456)$, then the vanishing of $(Q, Q')_2$ is the necessary and sufficient condition for the lines to be conjugate. We denote this expression (shorn of extraneous multiplicative factors) by $\mathcal{C}_{123}$ or $\mathcal{C}_{456}$; and similarly define the *Cayley-Salmon polynomial* $\mathcal{C}_\alpha \in \mathfrak{P}$ for each 3-element set $\alpha \subseteq$ SIX. (Thus, $\mathcal{C}_\alpha = \mathcal{C}_{\text{SIX}\backslash\alpha}$ by construction.) It turns out that each $\mathcal{C}_\alpha$ is homogeneous of degree 18; moreover, it is a product of six irreducible factors which exhibit a high degree of symmetry. We will describe the structure of $\mathcal{C}_{123}$ in detail, and then the action of the permutation group can be used to infer it for any $\mathcal{C}_\alpha$.

An explicit factorisation (done in MAPLE) shows that

$$\mathcal{C}_{123} = L_1\, L_2\, L_3\, M_4\, M_5\, M_6,$$

where each L_i and M_j is a homogeneous cubic in $a, \ldots, f$. (The factors are determined only up to nonzero multiplicative scalars, but this will cause no difficulty.) For instance,

$$L_1 = (c\,d\,e + b\,c\,d + a\,d\,f + a\,b\,e + a\,b\,f + b\,e\,f) - (a\,c\,d + b\,d\,e + c\,d\,f + b\,c\,e + a\,c\,f + a\,e\,f). \quad (4.1)$$

The following discussion will clarify how the remaining factors are thereby determined, and why they are so labelled.

4.3. The group $\mathfrak{S}(\text{LTR})$ acts naturally on $\mathfrak{P}$, and so does the group $\mathfrak{S}(\text{SIX})$ via ζ^{-1}. Let $G = G_{123} = G_{456}$ denote the subgroup of $\mathfrak{S}(\text{SIX})$ generated by $\mathfrak{S}(\{1,2,3\})$ and $\mathfrak{S}(\{4,5,6\})$, together with all elements of the form $(x\,x')\,(y\,y')\,(z\,z')$, where $\{x,y,z\} = \{1,2,3\}$ and $\{x',y',z'\} = \{4,5,6\}$. In other words, G is the subset of elements in $\mathfrak{S}(\text{SIX})$ which stabilise the unordered pair of sets $\{\{1,2,3\}, \{4,5,6\}\}$.

A direct calculation shows that G permutes the set of factors $\mathcal{F}_{123} = \{L_1, L_2, L_3, M_4, M_5, M_6\}$; specifically, we have a homomorphism $G \longrightarrow \mathfrak{S}(\mathcal{F}_{123})$ given by

$$(1\,2) \longrightarrow (L_1\,L_2), \qquad\qquad (1\,3) \longrightarrow (L_1\,L_3),$$
$$(4\,5) \longrightarrow (M_4\,M_5), \qquad\qquad (4\,6) \longrightarrow (M_4\,M_6),$$
$$(1\,4)\,(2\,5)\,(3\,6) \longrightarrow (L_1\,M_4)\,(L_2\,M_5)\,(L_3\,M_6).$$

This is interpreted as follows. Since $\zeta^{-1}(1\,2) = (\mathbb{A}\,\mathbb{E})\,(\mathbb{B}\,\mathbb{D})\,(\mathbb{C}\,\mathbb{F})$, the simultaneous substitution

$$a \to e, \quad e \to a, \quad b \to d, \quad d \to b, \quad c \to f, \quad f \to c,$$

interchanges L_1 and L_2. The same recipe applies throughout; for instance, since

$$\zeta^{-1}((1\,4)\,(2\,5)\,(3\,6)) = (\mathbb{A}\,\mathbb{B}),$$

the simultaneous substitution $a \to b, b \to a$ acts as the permutation $(L_1\,M_4)\,(L_2\,M_5)\,(L_3\,M_6)$.

The L_i and M_j are labelled so as to harmonise with the action of G. Since this action is transitive on $\mathcal{F}_{123}$, each of the remaining factors $L_2, \ldots, M_6$ is obtainable by a change of variables in L_1.

4.4. Invariance. Consider a fractional linear transformation (FLT)

$$\mu(z) = \frac{p\,z + q}{r\,z + s},$$

where p, q, r, s and z are, at the moment, seen as formal indeterminates. Then a calculation shows that

$$L_1(\mu(a), \ldots, \mu(f)) = \frac{(p\,s - q\,r)^3}{(r\,a + s) \ldots (r\,f + s)} \, L_1(a, \ldots, f).$$

In other words, L_1 is an invariant for the action of FLTs on ordered sextuples in $\mathbb{P}^1$. A general theory of such functions may be found in [9, Ch. 11.2] or [10, §1].

4.5. Given a hexad $\mathrm{LTR} \xrightarrow{h} \Gamma \subseteq \mathbf{C}$, one can evaluate each of the polynomials above by substituting $a = h(\mathbb{A}), \ldots, f = h(\mathbb{F})$. We will disallow the value ∞ for simplicity, although it would not have been difficult to account for this possibility.

Proposition 4.1. Assume that h is an alignment. Then $M_4 = M_5 = M_6 = 0$.

PROOF. By the invariance mentioned above, the vanishing is unaffected by an FLT of Γ, hence we may assume $a = 1, b = 0, c = -1$. If t denotes the common cross-ratio in (2.1), then we must have

$$d = \frac{1 - t}{1 + t}, \quad e = \frac{1}{2\,t - 1}, \quad f = \frac{t}{t - 2}.$$

Make a change of variables $a \to b, b \to a$ into L_1 and substitute the values above; then the expression is seen to vanish. This proves that $M_4 = 0$. Now recall that $\zeta^{-1}(4\,5) \in \Theta$, i.e., the substitution $\zeta^{-1}(4\,5) = (\mathbb{A}\,\mathbb{D})\,(\mathbb{B}\,\mathbb{E})\,(\mathbb{C}\,\mathbb{F})$ followed by h remains an alignment. This shows that $M_5 = 0$, and likewise for M_6. $\qquad\square$

4.6. Now assume $\alpha = \{1, 2, 4\}$. We have a factorisation

$$\mathcal{C}_{124} = L_1'\,L_2'\,L_4'\,M_3'\,M_5'\,M_6',$$

with similar properties. Define the group G_{124} analogously, and let $\mathcal{F}_{124} = \{L_1', L_2', L_4', M_3', M_5', M_6'\}$. As before, we have a homomorphism $G_{124} \longrightarrow \mathfrak{S}(\mathcal{F}_{124})$ given by

$$(1\,2) \longrightarrow (L_1'\,L_2'), \qquad\qquad (1\,4) \longrightarrow (L_1'\,L_4'),$$
$$(3\,5) \longrightarrow (M_3'\,M_5'), \qquad\qquad (3\,6) \longrightarrow (M_3'\,M_6'),$$
$$(1\,3)\,(2\,5)\,(4\,6) \longrightarrow (L_1'\,M_3')\,(L_2'\,M_5')\,(L_4'\,M_6').$$

The element $(3\,4)$ takes the set $\{1, 2, 3\}$ to $\{1, 2, 4\}$, and the change of variables induced by

$$(3\,4) \longrightarrow (\mathbb{A}\,\mathbb{E})\,(\mathbb{B}\,\mathbb{C})\,(\mathbb{D}\,\mathbb{F})$$

gives a bijection $\beta_{(3,4)} : \mathcal{F}_{123} \longrightarrow \mathcal{F}_{124}$ as follows:

$$L_1 \longrightarrow L_1', \quad L_2 \longrightarrow L_2', \quad L_3 \longrightarrow L_4',$$
$$M_4 \longrightarrow M_3', \quad M_5 \longrightarrow M_5', \quad M_6 \longrightarrow M_6'.$$

In this way, the structure of each $\mathcal{F}_\alpha$ is determined by that of $\mathcal{F}_{123}$. It follows that the invariance property is also true of each of the factors of $\mathcal{C}_\alpha$.

Lemma 4.2. The forms M_4 and M_3' are equal (up to a scalar).

PROOF. Let $u = (1\,4)\,(2\,5)\,(3\,6)$ and $v = (3\,4)$. Since $u(L_1) = M_4$ and $v(M_4) = M_3'$, it would suffice to show that $u^{-1}\,v\,u = (1\,6)$ stabilises L_1 up to a scalar. But $(1\,6) \longrightarrow (\mathbb{A}\,\mathbb{C})\,(\mathbb{B}\,\mathbb{E})\,(\mathbb{D}\,\mathbb{F})$, and it is easy to see that the corresponding change of variables turns L_1 into $-L_1$. $\qquad\square$

In particular, if h is an alignment, then M_3' vanishes and hence so does $\mathcal{C}_{124}$.

Proposition 4.3. Assume that Γ is a tri-symmetric sextuple. Then Γ satisfies CSC.

PROOF. Fix an alignment h, and consider a 3-element subset $\alpha \subseteq \text{SIX}$. By changing α to $\text{SIX}\setminus\alpha$ if necessary, we may assume that α has at least two elements in common with $\{1, 2, 3\}$. But then we are in the situation above up to a change of indices, hence $\mathcal{C}_\alpha = 0$. (This is best seen with an example. Say, $\alpha = \{2, 4, 6\}$, which we may change to $\{1, 3, 5\}$. Now the element $\zeta^{-1}((2\,3)\,(4\,5)) \in \Theta$ preserves $\mathcal{F}_{123}$ and induces a bijection between $\mathcal{F}_{124}$ and $\mathcal{F}_{135}$. Hence $\mathcal{C}_{135} = 0$.) $\qquad\square$

This proves the implication $(1) \Rightarrow (2)$ of the main theorem.

4.7. For the converse, assume that $\text{LTR} \xrightarrow{\ h\ } \Gamma$ is a hexad such that each $\mathcal{C}_\alpha$ is zero. Then (at least) one of the factors in $\mathcal{F}_{123}$ must vanish, and we may assume it to be L_1 after a change of variables. Let $a = 1, b = 0, c = -1$ after an FLT, then (4.1) reduces to

$$L_1 = d\,(1 - e + 2\,f) - f\,(1 + e) = 0. \tag{4.2}$$

Now (at least) one of the six factors in $\mathcal{F}_{124}$ must also vanish. When we pair each of them in turn with L_1, there are six cases to consider. First, assume that

$$L_1' = d + e - f - d\,e\,f = 0. \tag{4.3}$$

Now these two equations have a general solution

$$d = t, \quad e = \frac{2\,t^2}{1 + t^2}, \quad f = \frac{t\,(t + 1)}{2\,t^2 - t + 1};$$

for a parameter t. But then

$$\mathbb{A} \to 1, \quad \mathbb{B} \to 0, \quad \mathbb{C} \to t, \quad \mathbb{D} \to -1, \quad \mathbb{E} \to \frac{t\,(t + 1)}{2\,t^2 - t + 1}, \quad \mathbb{F} \to \frac{2\,t^2}{1 + t^2}$$

is an alignment, and hence Γ must be tri-symmetric. Replacing L_1' with L_4', M_5' or M_6' leads to similar solutions, and to the same conclusion.

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4.8. The remaining two cases behave a little differently. Assume that $L_1 = 0$ as above, and

$$L_2' = d - d\,e - d\,f + 2\,e\,f - d\,e\,f = 0.$$

The general solution is

$$d = \frac{t}{2+t}, \quad e = \frac{t-1}{t+3}, \quad f = t,$$

which by itself does not force Γ to be tri-symmetric. However, substituting it into $\mathcal{C}_{125}$ leads to the equation

$$\frac{(t-3)\,(3\,t-1)\,(3\,t+1)\,(t^2+1)^2}{(t+2)^{10}\,(t+3)^{11}} = 0.$$

Hence t can only be $3, \pm\frac{1}{3}$ or $\pm i$. Now in fact Γ is tri-symmetric for each of these special values; for instance, if $t = \frac{1}{3}$ then

$$\mathbb{A} \to 1, \quad \mathbb{B} \to \frac{1}{7}, \quad \mathbb{C} \to -\frac{1}{5}, \quad \mathbb{D} \to 0, \quad \mathbb{E} \to \frac{1}{3}, \quad \mathbb{F} \to -1$$

becomes an alignment. Replacing L_2' by M_3' leads to the same inference by a similar route. In conclusion, the conditions $\mathcal{C}_{123} = \mathcal{C}_{124} = \mathcal{C}_{125} = 0$ already force Γ to be tri-symmetric. The main theorem is now completely proved. $\qquad\square$

In summary, we have an assembly of homomorphisms

$$G_\alpha \longrightarrow \mathfrak{S}(\mathcal{F}_\alpha), \quad \alpha \subseteq \text{SIX};$$

and the point of the theorem lies in the interconnections between them. It is an intricate and highly regular structure on the whole, but at the moment I see no way of gaining any insight into it except by a direct computational attack as above. It would be of interest to have a more conceptual explanation for the entire phenomenon.

4.9. The undefined g-lines. Some of the g-lines may become undefined on a tri-symmetric sextuple, i.e., the corresponding quadratic forms may vanish identically. It is easy to classify all such cases by a direct computation. Assume Γ to be as in §1.5.

- The line $g(456)$ is undefined for any p; in fact all the Kirkman points $K[w, 456], w = 1, 2, 3$ coincide with the pole of the line $Q_4 Q_5 Q_6$ in Figure 2.
- Additionally, if $p = \pm i$ then $g(\alpha)$ is undefined for every 3-element subset $\alpha \subseteq \{3, 4, 5, 6\}$.

For a general p, the g-lines which remain defined are not all distinct. We have

$$g(145) = g(245) = g(345)$$

and similarly with 45 replaced by 46 and 56. Thus there are only 13 distinct lines.

When $p = \pm i$, we have $g(134) = g(234)$ and similarly with 34 replaced by $35, 36, 45, 46, 56$. Thus there are only 10 distinct lines.

4.10. We mention another characterisation of tri-symmetry. Given a triangle $\Delta = PQR$ in $\mathbb{P}^2$, let P' denote the pole of the line QR with respect to $\mathcal{K}$, and similarly for Q', R'. Then $\Delta' = P'Q'R'$ is called the polar triangle of Δ. It is a theorem due to Chasles (see [7, §7]) that these two triangles are in perspective, i.e., the lines PP', QQ', RR' are concurrent, say in a point $\tau(\Delta) = \tau(\Delta')$. It is also the case that the three points

$$PQ \cap P'Q', \quad PR \cap P'R', \quad QR \cap Q'R'$$

lie on the polar line of this point. In other words, the two triangles are in Desargues configuration in such a way that the point of perspectivity and the axis of perspectivity are in pole-polar relation.

Proposition 4.4. A sextuple Γ is tri-symmetric, if and only if it can be decomposed into two triangles Δ_1, Δ_2 such that $\tau(\Delta_1) = \tau(\Delta_2)$.

PROOF. Assume Γ to be tri-symmetric. With $A, \ldots, F$ as in §2.3, a straightforward calculation shows that $\tau(ABC)$ and $\tau(DEF)$ are both represented by the form

$$T = x_1^2 - x_1 x_2 + x_2^2.$$

As to the converse, let $\Gamma = \Delta_1 \cup \Delta_2$ be a decomposition as above. Since the construction of τ is compatible with automorphisms of $\mathcal{K}$, we may assume that Δ_1 has vertices $0, 1, \infty$ so that $\tau(\Delta_1)$ is given by T. If Δ_2 has vertices u, v, w, then Δ_2' corresponds to

$$(x_1 - v\, x_2)\,(x_1 - w\, x_2), \quad (x_1 - u\, x_2)\,(x_1 - w\, x_2), \quad (x_1 - u\, x_2)\,(x_1 - v\, x_2).$$

Since $\tau(\Delta_2')$ is also given by T, the three quadratic forms

$$(x_1 - u\, x_2)^2, \quad (x_1 - v\, x_2)\,(x_1 - w\, x_2), \quad T$$

must be linearly dependent and hence the 3×3 matrix of their coefficients must have zero determinant. The same holds for the other two vertex pairs. This reduces to a set of equations

$$\beta^2 - \alpha\,\beta - \alpha + 3\,\beta + 1 = u\,\alpha - u - \alpha + \beta + 3 = u\,\beta + 1 = 0,$$

where $\alpha = v + w$ and $\beta = v\,w$. The solutions are

$$\alpha = \frac{u^2 - 3\,u + 1}{u\,(u-1)}, \quad \beta = -\frac{1}{u},$$

implying that $\{v, w\} = \{\frac{u-1}{u}, \frac{1}{1-u}\}$. Hence $\Gamma = \{0, 1, \infty, u, \frac{u-1}{u}, \frac{1}{1-u}\}$ is tri-symmetric. $\qquad\square$

The line $Q_4 Q_5 Q_6$ in Figure 2 is represented by the form T (see [4, §4.5]), hence the point $\tau(ABC) = \tau(DEF)$ is the pole of this line. It is not shown there explicitly, but in any event must lie in the interior of the conic.

5. THE IDEAL OF THE TRI-SYMMETRIC LOCUS

In this section we will determine the SL_2-equivariant minimal resolution of the defining ideal of Ω using some elimination-theoretic computations. The reader is referred to [5] for a discussion of the invariant theory of binary forms. (The set-up used there involves binary quintics rather than sextics, but the general formalism is identical.) An exposition of the same material may also be found in [22, Ch. 4].

5.1. Let

$$\mathfrak{f} = \sum_{i=0}^{6} a_i \binom{6}{i} x_1^{6-i} x_2^i,$$

denote the generic binary sextic with indeterminate coefficients, and let

$$R = \mathbf{C}[a_0, \ldots, a_6] = \mathrm{Sym}^\bullet(S_6),$$

denote the coordinate ring of $\mathbb{P}^6$. Write

$$\Psi_p = x_1\, x_2\, (x_1 - x_2)\, (x_1 - p\, x_2)\, (x_1 - \frac{p-1}{p}\, x_2)\, (x_1 - \frac{1}{1-p}\, x_2),$$

for the sextic form representing a tri-symmetric sextuple as in §1.5.

Let $J \subseteq R$ denote the homogeneous defining ideal of Ω; it consists of forms which vanish on the union of SL_2-orbits of Ψ_p taken over all p. In order to calculate it, choose indeterminates $\alpha, \beta, \gamma, \delta$, and make substitutions

$$x_1 \to \alpha\, y_1 + \beta\, y_2, \quad x_2 \to \gamma\, y_1 + \delta\, y_2,$$

into $(p-1)\, p\, \Psi_p$ to get a new sextic $\mathfrak{g}(y_1, y_2)$. Write it as

$$\mathfrak{g}(y_1, y_2) = \sum_{i=0}^{6} \binom{6}{i} \varphi_i\, y_1^{6-i}\, y_2^i,$$

where φ_i are polynomial expressions in the variables $\alpha, \beta, \gamma, \delta, p$. This defines a ring homomorphism

$$\mathbf{C}[a_0, \ldots, a_6] \xrightarrow{g} \mathbf{C}[\alpha, \beta, \gamma, \delta, p], \qquad a_i \longrightarrow \varphi_i,$$

so that $J = \ker(g)$. I have calculated this ideal explicitly in MACAULAY-2; one pleasant surprise is that it turns out to be a perfect ideal with minimal resolution

$$0 \leftarrow R/J \leftarrow R \leftarrow R(-5)^5 \leftarrow R(-6)^3 \oplus R(-7) \leftarrow 0.$$

In other words, Ω is an arithmetically Cohen-Macaulay subvariety. Since Ω has Hilbert polynomial

$$\frac{2}{3}\, t^4 - \frac{5}{6}\, t^3 + \frac{35}{6}\, t^2 - \frac{2}{3}\, t + 2,$$

its degree is $\frac{2}{3} \times 4! = 16$, in agreement with Proposition 3.2.

We should like to identify the SL_2-representation corresponding to the 5-dimensional Betti module of minimal generators in degree 5. It must be a subrepresentation of $R_5 = \mathrm{Sym}^5(S_6)$. By the Cayley-Sylvester formula,

$$R_5 = S_2 \oplus S_2 \oplus S_4 \oplus \{\text{summands of dimensions} > 5\},$$

and hence on dimensional grounds the module can only be S_4. In other words, J is minimally generated by the coefficients of a covariant of degree-order $(5, 4)$. The complete minimal system for binary sextics is given in [11, p. 156]; it shows that there is a unique such covariant up to a scalar. It can described as follows: define

$$\vartheta_{24} = (\mathfrak{f}, \mathfrak{f})_4, \quad \vartheta_{32} = (\vartheta_{24}, \mathfrak{f})_4, \quad \vartheta_{54} = (\vartheta_{24}, \vartheta_{32})_1,$$

where ϑ_{mq} is of degree-order (m, q). Thus J must be generated by the coefficients of ϑ_{54}. This implies that $\Gamma = \{z_1, \ldots, z_6\}$ is tri-symmetric exactly when the covariant ϑ_{54} vanishes identically on the sextic $\prod_{i=1}^{6}(x_1 - z_i x_2)$.

The resolution shows that J has a three-dimensional space of linear syzygies, and a unique quadratic syzygy. In order to find the former, we must look for an identical relation of the form $(\vartheta_{54}, \mathfrak{f})_r = 0$. Since binary sextics has no covariant of degree-order $(6, 2)$, perforce $(\vartheta_{54}, \mathfrak{f})_4 = 0$. The quadratic syzygy is similarly accounted for by the identity $(\vartheta_{54}, \vartheta_{24})_4 = 0$, since there is no invariant in degree 7. In summary, the equivariant minimal resolution of J is

$$0 \leftarrow R/J \leftarrow R \leftarrow R(-5) \otimes S_4 \leftarrow R(-6) \otimes S_2 \oplus R(-7) \leftarrow 0.$$

5.2. Since ϑ_{54} is defined to be the Jacobian of ϑ_{24} and ϑ_{32}, its vanishing implies a functional dependency between the latter two. This is confirmed by a direct computation. Indeed,

$$\vartheta_{24}(\Psi_p) = f_1(p)\, T^2, \quad \text{and} \quad \vartheta_{32}(\Psi_p) = f_2(p)\, T,$$

where

$$f_1(p) = \frac{(p^2 + 1)\,(p^2 - 2p + 2)\,(2p^2 - 2p + 1)}{75\,p^2\,(p-1)^2}, \qquad f_2(p) = -\frac{(p-2)\,(2p-1)\,(p+1)}{30\,p\,(p-1)}\, f_1(p)$$

are rational functions of p, and

$$T = x_1^2 - x_1 x_2 + x_2^2,$$

as in §4.10. Thus, when $\mathfrak{f}$ is specialised to a tri-symmetric sextuple, ϑ_{32} evaluates to a quadratic form which corresponds to the line containing its three centres of involution.

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5.3. A special case deserves to be mentioned. Notice that $f_1(p)$ vanishes for

$$p = \pm i,\ 1 \pm i,\ \frac{1}{2}\,(1 \pm i). \tag{5.1}$$

Thus ϑ_{24} and ϑ_{32} are both identically zero for these values of p, suggesting that the line containing the three centres becomes 'indeterminate' in some sense. As we will see, this is indeed so.

Specialise to $p = i$, i.e., let

$$A = 0, \quad B = 1, \quad C = \infty, \quad D = i, \quad E = \frac{i-1}{i} = 1 + i, \quad F = \frac{1}{1-i} = \frac{1}{2}(1 + i).$$

We have already seen in §4.10 that $\underbrace{\tau(ABC) = \tau(DEF)}_{T_1}$ for any p. However, a direct calculation shows that in this case we additionally have

$$\underbrace{\tau(ADF) = \tau(BCE)}_{T_2}, \quad \underbrace{\tau(ACD) = \tau(BEF)}_{T_3}, \quad \underbrace{\tau(ABF) = \tau(CDE)}_{T_4}.$$

Thus Γ has altogether twelve centres of involution, which lie by threes on four lines.

The group $\Theta(i)$ is strictly larger than the generic case. It is easily seen that $(\mathbb{A}\,\mathbb{E})$ and $(\mathbb{C}\,\mathbb{F})$ are in $\Theta(i)$; either element will turn all cross-ratios in (3.1) into $-i$. Then it follows automatically that $(\mathbb{B}\,\mathbb{D}) \in \Theta(i)$. If V denotes the group generated by these two transpositions (so that V is isomorphic to the Klein four-group), then $\Theta(i)$ is the internal direct product of Θ and V. Hence $|\Theta(i)| = 144$.

The group V permutes the four lines; for instance, transposing A and E will turn the pair of triangles ABC, DEF into BCE, ADF. The morphism $V \longrightarrow \mathfrak{S}(\{T_1, T_2, T_3, T_4\})$ is given by

$$(\mathbb{A}\,\mathbb{E}) \longrightarrow (T_1\,T_2)\,(T_3\,T_4), \quad (\mathbb{C}\,\mathbb{F}) \longrightarrow (T_1\,T_4)\,(T_2\,T_3).$$

5.4. In fact all the forms Ψ_p for any of the six values given in (5.1) lie in the same SL_2-orbit in $\mathbb{P}^6$, which we denote by $\mathcal{Z}$. This is a well-known geometric object. The orbit is Zariski closed in $\mathbb{P}^6$; indeed there are very few orbits of binary forms which have this property, and they have all been classified in [1]. (There it is described as the orbit of $x_1^5\,x_2 - x_1\,x_2^5$, which comes to the same thing.) Moreover, $\mathcal{Z} \subseteq \mathbb{P}^6$ is an arithmetically Gorenstein subvariety in codimension 3, and its ideal I is generated by the coefficients of ϑ_{24} (see [15, §3]). It has a self-dual Buchsbaum-Eisenbud resolution

$$0 \leftarrow R/I \leftarrow R \leftarrow R(-2) \otimes S_4 \leftarrow R(-3) \otimes S_4 \leftarrow R(-5) \leftarrow 0.$$

Unfortunately one cannot draw a diagram of this sextuple and its twelve centres of involution, since the equality $\langle A, B, C, F \rangle = i$ implies that not all six points can be chosen to be simultaneously real.

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5.5. Bolza's equations. In [3], Bolza classifies all binary sextics with nontrivial automorphism groups. The fourth canonical form on his list (loc. cit, page 50)

$$\mathfrak{B}_a = x_1^6 + a\, x_1^3\, x_2^3 + x_2^6, \qquad (a \text{ general})$$

turns out to be the same as Ψ_p, up to a linear transformation. Indeed, a direct calculation shows that

$$\vartheta_{24}(\mathfrak{B}_a) = \left(\frac{a^2}{25} + 2\right) x_1^2\, x_2^2, \qquad \vartheta_{32}(\mathfrak{B}_a) = \frac{a}{250}\, (a^2 + 50)\, x_1\, x_2,$$

and hence $\vartheta_{54}(\mathfrak{B}_a) = 0$. (The actual value of p is a little awkward to find; it is a root of the equation $\frac{f_2^2(p)}{f_1(p)} = \frac{a^2\,(a^2+50)}{2500}$.) Thus, the Zariski closure of the union of SL_2-orbits of $\mathfrak{B}_a$ is the same as Ω.

The group of linear automorphisms of $\mathfrak{B}_a$ (and hence that of Ψ_p) is called by Bolza the 'diedron (i.e., dihedral) group for $n = 3$', which is, of course, the same as the symmetric group on three objects. This gives another justification for the term 'tri-symmetric'. In the notation of Figure 2, it is generated by

$$\zeta^{-1}(5\,6) = (\mathbb{A}\,\mathbb{E})\,(\mathbb{C}\,\mathbb{D})\,(\mathbb{B}\,\mathbb{F}), \quad \zeta^{-1}(4\,6) = (\mathbb{A}\,\mathbb{F})\,(\mathbb{C}\,\mathbb{E})\,(\mathbb{B}\,\mathbb{D}), \quad \zeta^{-1}(4\,5) = (\mathbb{A}\,\mathbb{D})\,(\mathbb{B}\,\mathbb{E})\,(\mathbb{C}\,\mathbb{F}).$$

However, the group Θ from §3 is larger, since its elements do not necessarily come from linear transformations in the x_i.

According to the table on [3, p. 70], the variety Ω is set-theoretically defined by two invariant equations in degrees 10 and 12. They can be written down as follows: for $i = 2, 4, 6, 10$, let $\vartheta_{(i)}$ denote the degree i invariant of binary sextics (cf. [3, p. 51]). Now a binary sextic with distinct roots is tri-symmetric, exactly when

$$9\,\vartheta_{(10)} - 2\,\vartheta_{(4)}\,(6\,\vartheta_{(6)} + \vartheta_{(2)}\,\vartheta_{(4)}) = \vartheta_{(6)}^2 - \frac{1}{6}\,\vartheta_{(4)}^3 = 0.$$

The calculations in §5.1 show that both of these invariants are contained in the ideal generated by the coefficients of ϑ_{54}.

REFERENCES

[1] P. Aluffi and C. Faber. Linear orbits of d-tuples of points in $\mathbb{P}^1$. *J. Reine Angew. Math.*, vol. 445, pp. 205–220, 1993.

[2] H. F. Baker. *Principles of Geometry, vol. II*. Cambridge University Press, 1923.

[3] O. Bolza. On binary sextics with linear transformations into themselves. *Amer. J. of Math.*, vol. 10, no. 1, pp. 47–70, 1887.

[4] J. Chipalkatti. On the coincidences of Pascal lines. Available at `arXiv.org` (arXiv:1407.1447 [math.AG]). To appear in *Forum Geometricorum*.

[5] J. Chipalkatti. On Hermite's invariant for binary quintics. *J. Algebra*, vol. 317, no. 1, pp. 324–353, 2007.

[6] J. Conway and A. Ryba. The Pascal mysticum demystified. *Math. Intelligencer*, vol. 34, no. 3, pp. 4–8, 2012.

[7] H. S. M. Coxeter. *Projective Geometry*. Springer–Verlag, New York, 1987.

[8] H. S. M. Coxeter. Twelve points in $PG(5, 3)$ with 95040 self-transformations. Proc. Roy. Soc. London. Ser. A, vol. 247, pp. 279–293, 1958.

[9] I. Dolgachev. *Lectures on Invariant Theory*. London Mathematical Society Lecture Note Series, No. 296, Cambridge University Press, 2003.

[10] I. Dolgachev and D. Ortland. *Point Sets in Projective Spaces and Theta Functions*. Astérisque, no. 165, Société Mathématique de France, 1988.

[11] J. H. Grace and A. Young. *The Algebra of Invariants*. Reprinted by Chelsea Publishing Co., New York, 1962.

[12] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley-Interscience, 1994.

[13] B. Grünbaum. *Configurations of Points and Lines*. Graduate Studies in Mathematics, Amer. Math. Soc., 2009.

[14] O. Hesse. Ueber die Reciprocität der *Pascal-Steiner*schen und der *Kirkman-Cayley-Salmon*schen Sätze von dem Hexagrammum mysticum. J. für Mathematik, vol. 68, no. 3, 1868.

[15] S. Mukai and H. Umemura. Minimal rational threefolds. in *Algebraic Geometry, Tokyo/Kyoto, 1982*, Lecture Notes in Mathematics, No. 1016, Springer-Verlag, 1983.

[16] D. Pedoe. *Geometry, A Comprehensive Course*. Reprinted by Dover Publications, New York, 1988.

[17] H. W. Richmond. The figure formed from six points in space of four dimensions. *Math. Ann.*, vol. 53, pp. 161–176, 1900.

[18] G. Salmon. *A Treatise on Conic Sections.* Reprint of the 6th ed. by Chelsea Publishing Co., New York, 2005.

[19] A. Seidenberg. *Lectures in Projective Geometry*. D. Van Nostrand Company, New York, 1962.

[20] J. G. Semple and G. T. Kneebone. *Algebraic Projective Geometry.* Oxford at the Clarendon Press, 1952.

[21] K. G. C. von Staudt. Ueber die Steinerschen Gegenpunkte, welche durch zwei in eine Curve zweiter Ordnung beschriebene Dreiecke bestimmt sind. *Crelle's J.*, vol. 62, pp. 142–150, 1863.

[22] B. Sturmfels. *Algorithms in Invariant Theory*. Texts and Monographs in Symbolic Computation. Springer–Verlag, Wien–New York, 1993.

[23] J. J. Sylvester. 'Note on the . . . six-valued function of six letters', in his Collected Mathematical Papers, vol. II, p. 264, (also see vol. I, p. 92), Cambridge University Press, 1904–1912.

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Jaydeep Chipalkatti

Department of Mathematics

University of Manitoba

Winnipeg, MB R3T 2N2

Canada.

jaydeep.chipalkatti@umanitoba.ca